INTERNATIONAL JOURNAL FOR NUMERICAL AND ANALYTICAL METHODS IN GEOMECHANICS, VOL. 21, 863–881 (1997)

FINITE ELEMENT ANALYSIS OF STEADY-STATE NATURAL CONVECTION PROBLEMS IN FLUID-SATURATED POROUS MEDIA HEATED FROM BELOW

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SUMMARY

In this paper, a progressive asymptotic approach procedure is presented for solving the steady-state Horton–Rogers–Lapwood problem in a fluid-saturated porous medium. The Horton–Rogers–Lapwood problem possesses a bifurcation and, therefore, makes the direct use of conventional finite element methods difficult. Even if the Rayleigh number is high enough to drive the occurrence of natural convection in a fluid-saturated porous medium, the conventional methods will often produce a trivial non-convective solution. This difficulty can be overcome using the progressive asymptotic approach procedure associated with the finite element method. The method considers a series of modified Horton–Rogers–Lapwood problems in which gravity is assumed to tilt a small angle away from vertical. The main idea behind the progressive asymptotic approach procedure is that through solving a sequence of such modified problems with decreasing tilt, an accurate non-zero velocity solution to the Horton–Rogers–Lapwood problem can be obtained. This solution provides a very good initial prediction for the solution to the original Horton–Rogers–Lapwood problem so that the non-zero velocity solution can be successfully obtained when the tilted angle is set to zero. Comparison of numerical solutions with analytical ones to a benchmark problem of any rectangular geometry has demonstrated the usefulness of the present progressive asymptotic approach procedure. Finally, the procedure has been used to investigate the effect of basin shapes on natural convection of pore-fluid in a porous medium.

Int. J. Numer. Anal. Meth. Geomech., Vol. 21, 863–881 (1997)
(No. of Figures: 10 No. of Tables: 0 No. of Refs: 10)

Key words: progressive asymptotic approach; natural convection; porous media; bifurcation

1. INTRODUCTION

In a fluid-saturated porous medium, a change in medium temperature may lead to a change in the density of pore-fluid within the medium. This change can be considered as a buoyancy force term in the momentum equation to determine pore-fluid flow in the porous medium using the Boussinesq approximation model. The momentum equation used to describe pore-fluid flow in a porous medium is usually established using Darcy's law or its extensions. If a fluid-saturated porous medium has a horizontal layer geometry, and is heated uniformly from the bottom of the layer, there may exist a temperature difference between the top and bottom boundaries of the layer. Since the positive direction of the temperature gradient due to this temperature difference is

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CCC 0363–9061/97/120863–19$17.50
Received 5 November 1996
Revised 6 August 1997

opposite to that of the gravity acceleration, there is no natural convection for a small temperature gradient in the porous medium. In this case, heat energy is solely transferred from the high-temperature region (the bottom of the horizontal layer) to the low-temperature region (the top of the horizontal layer) by thermal conduction. However, if the temperature difference is large enough, it may trigger natural convection in the fluid-saturated porous medium. This problem was first treated analytically by Horton and Rogers[1] as well as Lapwood,[2] and is often called the Horton–Rogers–Lapwood problem.

This kind of natural convection problem has been found in many scientific and engineering fields. For example, in geoenvironmental engineering, the buried nuclear waste and industrial waste in a fluid-saturated porous medium may generate heat and result in a temperature gradient in the vertical direction. If the Rayleigh number, which is directly proportional to the temperature gradient, is equal to or greater than the critical Rayleigh number, natural convection will take place in the porous medium, so that the groundwater may be severely contaminated due to the pore-fluid flow circulation caused by the natural convection. In geophysics, there may exist a vertical temperature gradient in the earth's crust. If this temperature gradient is big enough, it will cause regional natural convection in the earth's crust. In this situation, the pore-fluid flow circulation due to the natural convection can dissolve soluble minerals in some part of a region and carry them to another part of the region. This is the mineralization problem in geophysics. Since a natural porous medium is often of a complicated geometry and composed of many different materials, numerical methods are always needed to solve the aforementioned problems.

From the mathematical point of view, the Horton–Rogers–Lapwood problem possesses a bifurcation. The linear stability theory based on the first-order perturbation is commonly used to solve this problem analytically and numerically (see Reference 3 and the references therein). However, as Joly *et al.* pointed out in a recent paper,[4] 'The linear stability theory, in which the non-linear term of the heat disturbance equation has been neglected, does not describe the amplitude of the resulting convection motion. The computed disturbances are correct only for infinitesimal amplitudes. Indeed, even if the form of convective motion obtained for low super-critical conditions is often quite similar to the critical disturbance, the nonlinear term may produce manifest differences, especially when strong constraints, such as impervious or adiabatic boundaries, are considered'. Since it is the amplitude and the form of natural convective motion that significantly affect or dominate the contaminant transport and mineralization in a fluid-saturated porous medium, there is a definite need for including the full nonlinear term of the energy equation in the finite element analysis.

From the finite element analysis point of view, the direct inclusion of the full nonlinear term of the energy equation in the steady-state Horton–Rogers–Lapwood problem would result in a formidable difficulty. The finite element method needs to deal with a highly nonlinear problem and often suffers difficulties in finding out the true non-zero velocity field in a fluid-saturated porous medium because the Horton–Rogers–Lapwood problem always has a zero solution for the velocity field of the pore-fluid. If the velocity field of the pore-fluid used at the beginning of an iteration method is not chosen appropriately, then the resulting finite element solution always tends to zero for the velocity field in a fluid-saturated porous medium. Although this difficulty can be circumvented by turning a steady-state problem into a transient one,[5] it is often unnecessary and computationally inefficient to obtain a steady-state solution from solving a transient problem. Therefore, it is still highly desirable to develop a numerical procedure to directly solve the steady-state Horton–Rogers–Lapwood problem. For this reason, a progressive asymptotic approach procedure is presented and used in this paper. The concept of an asymptotic approach

has been successfully applied to some fields of the finite element method. For instance, the h-adaptive mesh refinement[6] is based on this concept and can produce a satisfactory solution with the progressive reduction in the size of finite elements used in the analysis. The same concept was recently employed to obtain asymptotic solutions for natural frequencies of vibrating structures in a finite element analysis.[7–9] To solve the steady-state Horton–Rogers–Lapwood problem with the full nonlinear term of the energy equation included in the finite element analysis, the asymptotic approach concept needs to be used in another fashion. If the gravity acceleration is assumed to tilt at a small angle, α, in the Horton–Rogers–Lapwood problem, then a non-zero velocity field in a fluid-saturated porous medium may be found using the finite element method. The resulting non-zero velocity field can be used as the initial velocity field of the pore-fluid to solve the original Horton–Rogers–Lapwood problem with the tilted small angle being zero. Thus, two kinds of problems need to be progressively solved in the finite element analysis. One is the modified Horton–Rogers–Lapwood problem, in which the gravity acceleration is tilted a small angle, and another is the original Horton–Rogers–Lapwood problem. This forms two basic steps of the progressive asymptotic approach procedure proposed and used in this paper. Clearly, the basic idea behind the progressive asymptotic approach procedure is that when the small angle tilted by the gravity acceleration approaches zero, the modified Horton–Rogers–Lapwood problem asymptotically approaches the original one and as a result, a solution to the original Horton–Rogers–Lapwood problem can be obtained. Some examples will be considered in the context of the paper to examine the applicability of the proposed progressive asymptotic approach procedure when it is used to deal with steady-state natural convection problems in fluid-saturated porous media.

2. GOVERNING EQUATIONS OF THE PROBLEM AND THE FINITE ELEMENT FORMULATION

For a two-dimensional fluid-saturated porous medium, if Darcy's law is used to describe pore-fluid flow and the Boussinesq approximation is employed to describe a change in pore-fluid density due to a change in pore-fluid temperature, the governing equations of natural convection for incompressible fluid in a steady-state can be expressed as

$$u_{i,i} = 0 \tag{1}$$

$$u_i = \frac{1}{\mu} k_{ij}(-p_{,j} + \rho_f g_i) \tag{2}$$

$$(\rho_0 c_p)\, u_j T_{,j} = (\lambda^e_{ij} T_{,j})_{,i} \tag{3}$$

$$\rho_f = \rho_0[1 - \beta(T - T_0)] \tag{4}$$

$$\lambda^e_{ij} = \phi\lambda_{ij} + (1-\phi)\lambda^s_{ij} \tag{5}$$

where u_i is the velocity component in the x_i direction, p and T are pressure and temperature, ρ_0 and T_0 are the reference density of pore-fluid and the reference temperature of the medium, μ and c_p are the dynamic viscosity and specific heat of pore-fluid, λ_{ij} and λ^s_{ij} are the second-order thermal conductivity tensor for the pore-fluid and solid matrix in the porous medium, ϕ and β are

the porosity of the medium and the thermal volume expansion coefficient of pore-fluid; k_{ij} is the second-order permeability tensor of the medium and g_i is the acceleration due to gravity acceleration component in the x_i direction.

In order to simplify equations (1)–(3), the following dimensionless variables are defined:

$$x_i^* = \frac{x_i}{H}, \qquad T^* = \frac{T - T_0}{\Delta T}, \qquad u_i^* = \frac{H\rho_0 c_p}{\lambda_{e0}} u_i$$

$$p^* = \frac{k_h \rho_0 c_p}{\mu \lambda_{e0}}(p - p_0), \qquad k_{ij}^* = \frac{k_{ij}}{k_h}, \qquad \lambda_{ij}^{e*} = \frac{\lambda_{ij}^e}{\lambda_{e0}} \tag{6}$$

where x_i^* are the dimensionless coordinates; u_i^* is the dimensionless velocity component in the x_i direction; p^* and T^* are the dimensionless pressure and temperature; k_h is a reference medium permeability coefficient in the horizontal direction; λ_{e0} is a reference thermal conductivity coefficient of the porous medium; ΔT is the temperature difference between the bottom and top boundaries of the porous medium; H is a reference length and p_0 is the static pore-fluid pressure.

If the porous medium considered is orthotropic, in which the x_2 axis is upward in the vertical direction and is in coincidence with the principal direction of medium permeability as well as that of medium thermal conductivity, then $k_{ij} = 0$ and $\lambda_{ij} = 0$ $(i \neq j)$. In such a case, substituting the above dimensionless variables into equations (1)–(3) yields

$$u_{i,i}^* = 0 \tag{7}$$

$$u_i^* = k_{ii}^*(-p_{,i}^* + RaT^* e_i) \quad \text{(no summation for } i\text{)} \tag{8}$$

$$u_j^* T_{,j}^* = \lambda_{jj}^{e*} T_{,jj}^* \tag{9}$$

where $\mathbf{e}$ is a unit vector and $\mathbf{e} = e_1\mathbf{i} + e_2\mathbf{j}$ for a two-dimensional problem. If the gravity acceleration is assumed to tilt a small angle α away from the vertical direction, then $e_1 = \sin\alpha$ and $e_2 = \cos\alpha$. Ra is the Rayleigh number, defined as

$$Ra = \frac{(\rho_0 c_p)\rho_0 g \beta \Delta T k_h H}{\mu \lambda_{e0}} \tag{10}$$

By considering the dimensionless velocity, pressure and temperature as basic variables, equations (7)–(9) can be discretized using the conventional finite element method[10] and the resulting finite element formulation for a typical element can be expressed as

$$\begin{bmatrix} \mathbf{M}^\mathrm{e} & \mathbf{0} & -\mathbf{B}_1^\mathrm{e} & -\mathbf{A}_1^\mathrm{e} \\ \mathbf{0} & \mathbf{M}^\mathrm{e} & -\mathbf{B}_2^\mathrm{e} & -\mathbf{A}_2^\mathrm{e} \\ \mathbf{0} & \mathbf{0} & \mathbf{E}^\mathrm{e} & \mathbf{0} \\ \mathbf{C}_1^\mathrm{e} & \mathbf{C}_2^\mathrm{e} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{U}_1^\mathrm{e} \\ \mathbf{U}_2^\mathrm{e} \\ \mathbf{T}^\mathrm{e} \\ \mathbf{P}^\mathrm{e} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_1^\mathrm{e} \\ \mathbf{F}_2^\mathrm{e} \\ \mathbf{G}^\mathrm{e} \\ \mathbf{0} \end{Bmatrix} \tag{11}$$

where $\mathbf{U}_1^\mathrm{e}$ and $\mathbf{U}_2^\mathrm{e}$ are the nodal dimensionless velocity vectors of the element in the x_1 and x_2 directions, respectively, $\mathbf{T}^\mathrm{e}$ and $\mathbf{P}^\mathrm{e}$ are the nodal dimensionless temperature and pressure vectors of the element, $\mathbf{A}_i^\mathrm{e}$, $\mathbf{B}_i^\mathrm{e}$, $\mathbf{C}_i^\mathrm{e}$, $\mathbf{E}^\mathrm{e}$ and $\mathbf{M}^\mathrm{e}$ are the property matrices of the element, $\mathbf{F}_i^\mathrm{e}$ and $\mathbf{G}^\mathrm{e}$ are the dimensionless nodal load vectors due to the dimensionless stress and heat

flux on the boundary of the element. These matrices and vectors can be derived and expressed as follows:

$$\mathbf{A}_i^e = \int_A \varphi_{,i} k_{ii}^* \mathbf{\Psi}^T \, dA, \qquad \mathbf{B}_i^e = \int_A \varphi k_{ii}^* Ra \varphi^T e_i \, dA, \qquad \mathbf{C}_i^e = \int_A \mathbf{\Psi} \varphi_{,i}^T \, dA$$

$$\mathbf{D}_i^e(u_i^*) = \int_A \varphi u_i^* \varphi_{,i}^T \, dA, \qquad \mathbf{L}_i^e = \int_A \varphi_{,i} \lambda_{ii}^* \varphi_{,i}^T \, dA, \qquad \mathbf{E}^e = \mathbf{D}_i^e(u_i^*) + \mathbf{L}_i^e$$

$$\mathbf{F}_i^e = k_{ii}^* \int_S \sigma_i^* \varphi \, dS, \qquad \mathbf{G}^e = -\int_S q^* \varphi \, dS, \qquad \mathbf{M}^e = \int_A \varphi \varphi^T \, dA$$

$$q^* = \frac{H}{\Delta T \lambda_{e0}} q, \qquad \sigma_i^* = \frac{k_h \rho_0 c_p}{\mu \lambda_{e0}} \sigma_i \tag{12}$$

where φ is the shape function vector of the temperature and velocity components of the element; Ψ is the shape function for the pressure of the element; σ_i and q are the stresses and heat flux on the boundary of the element; A and S are the area and boundary length of the element.

It is noted that since the full non-linear term of the energy equation in the Horton–Rogers–Lapwood problem is considered in the finite element analysis, matrix $\mathbf{E}^e$ is dependent on the velocity components of the element. Thus, a prediction for the initial velocities of an element is needed to get this matrix evaluated. Based on the progressive asymptotic approach procedure stated in the Introduction, the modified Horton–Rogers–Lapwood problem needs to be solved so as to give a very good prediction for the initial velocities of all elements in the system.

If the penalty finite element approach[10] is adopted, the pressure variable can be eliminated so that equation (11) is rewritten into the following form:

$$\begin{bmatrix} \mathbf{Q}^e & -\mathbf{B}^e \\ \mathbf{0} & \mathbf{E}^e \end{bmatrix} \begin{Bmatrix} \mathbf{U}^e \\ \mathbf{T}^e \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^e \\ \mathbf{G}^e \end{Bmatrix} \tag{13}$$

where

$$\mathbf{Q}^e = \bar{\mathbf{M}}^e + \frac{1}{\varepsilon} \mathbf{A}^e (\mathbf{M}_p^e)^{-1} (\mathbf{C}^e)^T$$

$$\bar{\mathbf{M}}^e = \begin{bmatrix} \mathbf{M}^e & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^e \end{bmatrix}, \qquad \mathbf{U}^e = \begin{Bmatrix} \mathbf{U}_1^e \\ \mathbf{U}_2^e \end{Bmatrix}, \qquad \mathbf{F}^e = \begin{Bmatrix} \mathbf{F}_1^e \\ \mathbf{F}_2^e \end{Bmatrix}$$

$$\mathbf{B}^e = \begin{Bmatrix} \mathbf{B}_1^e \\ \mathbf{B}_2^e \end{Bmatrix}, \quad \mathbf{A}^e = \begin{Bmatrix} \mathbf{A}_1^e \\ \mathbf{A}_2^e \end{Bmatrix}, \quad \mathbf{C}^e = \begin{Bmatrix} \mathbf{C}_1^e \\ \mathbf{C}_2^e \end{Bmatrix}$$

$$\mathbf{M}_p^e = \int_A \mathbf{\Psi} \mathbf{\Psi}^T \, dA \tag{14}$$

It needs to be pointed out that ε is a penalty parameter in equation (14). For the purpose of obtaining an accurate solution, this parameter must be chosen small enough to approximate fluid incompressibility well, but big enough to prevent the resulting matrix problem from becoming too ill-conditioned to solve.

By assembling all elements in a system, the finite element equation of the system can be expressed in a matrix form as

$$\begin{bmatrix} \mathbf{Q} & -\mathbf{B} \\ \mathbf{0} & \mathbf{E}(\mathbf{U}) \end{bmatrix} \begin{Bmatrix} \mathbf{U} \\ \mathbf{T} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{G} \end{Bmatrix} \tag{15}$$

where **Q**, **B** and **E** are global property matrices of the system; **U** and **T** are global nodal velocity and temperature vectors of the system; **F** and **G** are global nodal load vectors of the system. Since equation (15) is non-linear, either the successive substitution method or the Newton–Raphson method will be used to solve this equation.

3. PROGRESSIVE ASYMPTOTIC APPROACH PROCEDURE FOR SOLVING THE HORTON–ROGERS–LAPWOOD PROBLEM

As mentioned in the previous sections, if the full non-linear term in the energy equation is included in the finite element analysis, it is necessary to predict the initial velocity field of pore-fluid *a priori*. Thus, the key issue of obtaining a non-zero solution for the Horton–Rogers–Lapwood problem is to choose the initial velocity field of pore-fluid correctly. If the initial velocity field is not correctly chosen, the finite element method will lead to a zero solution for natural convection of pore-fluid, even though the Rayleigh number is high enough to drive the occurrence of natural convection in a fluid-saturated porous medium. In order to overcome this difficulty, a modified Horton–Rogers–Lapwood problem, in which the gravity acceleration is assumed to tilt a small angle α, needs to be solved. Supposing the non-zero solution for the modified Horton–Rogers–Lapwood problem is $S(\alpha)$, it is possible to find a non-zero solution for the original Horton–Rogers–Lapwood problem by taking a limit of $S(\alpha)$ when α approaches zero. This process can be mathematically expressed as follows:

$$\lim_{\alpha \to 0} S(\alpha) = S(0) \tag{16}$$

where $S(0)$ is a solution for the original Horton–Rogers–Lapwood problem, $S(\alpha)$ is the solution for the modified Horton–Rogers–Lapwood problem, S is any variable to be solved in the original Horton–Rogers–Lapwood problem.

It is noted that in theory, if $S(\alpha)$ could be expressed as a function of α explicitly, $S(0)$ would follow immediately. However, in practice, it is necessary to find out $S(0)$ numerically since it is very difficult and often impossible to express $S(\alpha)$ in an explicit manner. Thus, the question which must be answered is how to choose α so as to obtain an accurate non-zero solution $S(0)$. From the theoretical point of view, it is desirable to choose α as small as possible. The reason for this is that the smaller the value of α, the closer the characteristic of $S(\alpha)$ to that of $S(0)$. This enables a more accurate solution $S(0)$ to be obtained in the computation. From the finite element analysis point of view, α cannot be chosen too small because the smaller the value of α, the more sensitive the solution $S(\alpha)$ to the initial velocity field of pore-fluid. As a result, a very small α usually leads to a zero velocity field due to any inappropriate choice for the initial velocity field of pore-fluid. To avoid this phenomenon, α should be chosen big enough to eliminate the strong dependence of $S(\alpha)$ on the initial velocity field of pore-fluid. For the purpose of using a big value of α and keeping the final solution $S(0)$ of good accuracy in the finite element analysis, $S(\alpha)$ needs to approach $S(0)$

in a progressive asymptotic manner. This leads to the following processes:

$$\lim_{\alpha_i \to \alpha_{i+1}} S(\alpha_i) = S(\alpha_{i+1}) \quad (i = 1, 2, \ldots, n-1)$$
$$\lim_{\alpha_n \to 0} S(\alpha_n) = S(0)$$
$$\alpha_1 = \alpha, \qquad \alpha_{i+1} = \frac{1}{R}\alpha_i \tag{17}$$

where n is the total step number for α approaching zero, R is the rate of α_i approaching α_{i+1}. Generally, the values of α, n and R are dependent on the nature of a problem to be analysed.

For solving the steady-state Horton–Rogers–Lapwood problem using the progressive asymptotic approach procedure associated with the finite element method, numerical experience has shown that $1^\circ \leqslant \alpha \leqslant 5^\circ$, $5 \leqslant R \leqslant 10$ and $1 \leqslant n \leqslant 2$ leads to acceptable solutions. Therefore, for α being in the range of 1° to 5° and R being in the range of 5–10, $S(\alpha)$ can asymptotically approach $S(0)$ in one step or two steps. This indicates the efficiency of the present procedure.

4. DERIVATION OF ANALYTICAL SOLUTION TO A BENCHMARK PROBLEM

In order to verify the appplicability of the progressive asymptotic approach procedure for solving the Horton–Rogers–Lapwood convection problem, an analytical solution is needed for a benchmark problem, the geometry and boundary conditions of which can be exactly modelled by the finite element method. Although the existing solutions[3] for a horizontal layer in porous media can be used to check the accuracy of a finite element solution within a square box with appropriate boundary conditions, it is highly desirable to examine the progressive asymptotic approach procedure as extensively as possible. For this purpose, a benchmark problem of any rectangular geometry is constructed and shown in Figure 1. Without loss of generality, the dimensionless governing equations given in equations (7)–(9) are considered in this section. The boundary conditions of the problem are expressed using the dimensionless

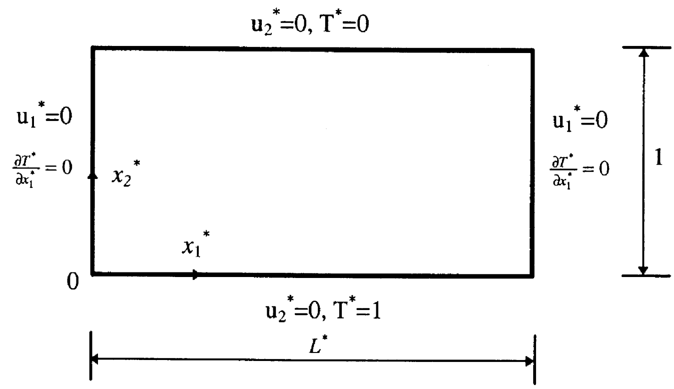


Figure 1. Geometry and boundary conditions of a benchmark problem

variables as follows:

$$u_1^* = 0, \qquad T_{,1}^* = 0 \quad (\text{at } x_1^* = 0 \text{ and } x_1^* = L^*)$$

$$u_2^* = 0, \qquad T^* = 1 \quad (\text{at } x_2^* = 0)$$

$$u_2^* = 0, \qquad T^* = 0 \quad (\text{at } x_2^* = 1) \tag{18}$$

where L^* is a dimensionless length in the horizontal direction and $L^* = L/H$, in which L is the real length of the problem domain in the horizontal direction.

For ease of deriving an analytical solution to the benchmark problem, it is assumed that the porous medium under cosideration is fluid-saturated and isotropic. This enables equations (7), (8) and (9) to be simplified as

$$u_{i,i}^* = 0 \tag{19}$$

$$u_i^* = -p_{,i}^* + RaT^* e_i \quad (\text{no summation for } i) \tag{20}$$

$$u_j^* T_{,j}^* = T_{,jj}^* \tag{21}$$

Using the linearization procedure for temperature gradient and a stream function Ψ simultaneously, equations (19)–(21) are reduced into the following two equations:

$$\Psi_{,ii} = -RaT_{,1}^* \tag{22}$$

$$\Psi_{,1} = T_{,jj}^* \tag{23}$$

Since equations (22) and (23) are linear ones, solutions to Ψ and T^* are of the following forms:

$$\Psi = f(x_2^*)\sin\left(q\frac{x_1^*}{L^*}\right) \quad (q = m\pi, m = 1, 2, 3, \ldots) \tag{24}$$

$$T^* = \theta(x_2^*)\cos\left(q\frac{x_1^*}{L^*}\right) + (1 - x_2^*) \quad (q = m\pi, m = 1, 2, 3, \ldots) \tag{25}$$

Substituting equations (24) and (25) into equations (22) and (23) yields the following equations:

$$f''(x_2^*) - \left(\frac{q}{L^*}\right)^2 f(x_2^*) = \frac{q}{L^*} Ra\theta(x_2^*) \tag{26}$$

$$\frac{q}{L^*} f(x_2^*) = -\left(\frac{q}{L^*}\right)^2 \theta(x_2^*) + \theta''(x_2^*) \tag{27}$$

Combining equation (26) and equation (27) leads to an equation containing $f(x_2^*)$ only:

$$f^{\mathrm{IV}}(x_2^*) - 2\left(\frac{q}{L^*}\right)^2 f''(x_2^*) - \left(\frac{q}{L^*}\right)^2 \left[Ra - \left(\frac{q}{L^*}\right)^2\right] f(x_2^*) = 0 \tag{28}$$

It is immediately noted that equation (28) is a linear, homogeneous ordinary differentiation equation so that it has a zero solution. For the purpose of finding out a non-zero solution, it is noted that the non-zero solution satisfying both equation (28) and the boundary conditions in equation (18) can be expressed as

$$f(x_2^*) = \sin(rx_2^*) \quad (r = n\pi, n = 1, 2, 3, \dots) \tag{29}$$

Using this equation, the condition under which the non-zero solution exists for equation (28) is derived and expressed as

$$Ra = \left(\frac{L^*}{q} r^2 + \frac{q}{L^*}\right)^2 = \left(\frac{n^2}{m} L^* + \frac{m}{L^*}\right)^2 \pi^2 \quad (m = 1, 2, 3, \dots, n = 1, 2, 3, \dots) \tag{30}$$

It can be observed from equation (30) that in the case of L^* being an integer, the minimum Rayleigh number is $4\pi^2$, which occurs when $n = 1$ and $m = L^*$. However, if L^* is not an integer, the minimum Rayleigh number is $(L^* + 1/L^*)^2\pi^2$, which occurs when $m = 1$ and $n = 1$. Since the minimum Rayleigh number determines the onset of natural convection in the fluid-saturated porous medium for the Horton–Rogers–Lapwood problem, it is often named as the critical Rayleigh number, Ra_{critical}.

For this benchmark problem, the mode shapes for the stream function and related dimensionless variables in correspondence to the critical Rayleigh number can be derived and expressed as follows:

$$\Psi = C_1 \sin\left(\frac{m\pi}{L^*} x_1^*\right) \sin(n\pi x_2^*)$$

$$u_1^* = n\pi C_1 \sin\left(\frac{m\pi}{L^*} x_1^*\right) \cos(n\pi x_2^*)$$

$$u_2^* = -\frac{m\pi}{L^*} C_1 \cos\left(\frac{m\pi}{L^*} x_1^*\right) \sin(n\pi x_2^*)$$

$$T^* = -\frac{C_1}{\sqrt{Ra_{\text{critical}}}} \cos\left(\frac{m\pi}{L^*} x_1^*\right) \sin(n\pi x_2^*) + (1 - x_2^*)$$

$$p^* = \frac{nL^*}{m} C_1 \cos\left(\frac{m\pi}{L^*} x_1^*\right) \cos(n\pi x_2^*) - \frac{Ra_{\text{critical}}}{2}(1 - x_2^*)^2 + C_2 \tag{31}$$

where the values of m, n and Ra_{critical} are dependent on whether L^* is an integer or not, C_1 is a non-zero constant and C_2 is an arbitrary constant. It is interesting to note that since Ra_{critical} is a function of L^*, it varies with a non-integer L^*. This implies that if rectangular valleys are filled with porous media, they may have different critical Rayleigh numbers when their ratios of length to height are different.

5. VERIFICATION AND APPLICATION OF THE PROPOSED ASYMPTOTIC APPROACH PROCEDURE ASSOCIATED WITH FINITE ELEMENT ANALYSIS

Using the analytical solution derived for a benchmark problem in the last section, the proposed progressive asymptotic approach procedure associated with the finite element analysis for solving the Horton–Rogers–Lapwood problem in a fluid-saturated porous medium is validated in this section. A rectangular domain of $L^* = 1.5$ is considered in the calculation. The critical Rayleigh number for the test problem considered is $169/36\pi^2$. As shown in Figure 2, the problem domain is discretized into 576 nine-node quadrilateral elements of 2401 nodes in total. The mesh gradation technique, which enables the region in the vicinity of problem boundaries to be modelled using finite elements of small sizes, has been employed to increase the solution accuracy in this region. The following parameters associated with the progressive asymptotic approach procedure are used in the calculation: $\alpha = 5°$, $n = 2$ and $R = 5$.

Figures 3 to 6 show the comparison of numerical solutions with analytical ones for dimensionless velocity, stream function, temperature and pressure modes, respectively. In these figures, the plots in the above are analytical solutions, whereas the plots in the below are numerical solutions for the problem. It is observed from these results that the numerical solutions from the progressive asymptotic approach procedure associated with the finite element method are in good agreement with the analytical solutions. Compared with the analytical solutions, the maximum error in the numerical solutions is less than 2%. This demonstrates the usefulness of the present progressive asymptotic approach procedure when it is used to solve the steady-state Horton–Rogers–Lapwood problems.

At this point, there is a need to explain why both the analytical and the numerical solutions for the pore-fluid flow are non-symmetric, although the geometry and boundary conditions for the problem are symmetric. As stated in the Introduction, the Horton–Rogers–Lapwood problem belongs mathematically to a bifurcation problem. The trivial solution for the pore-fluid flow of

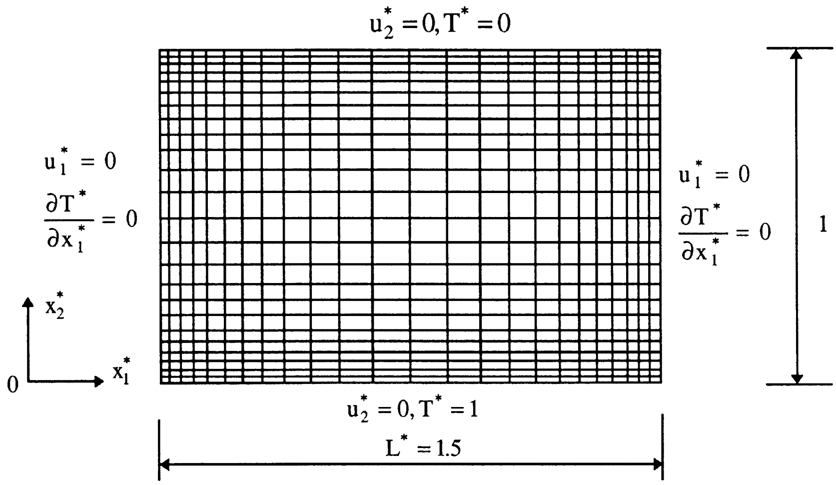
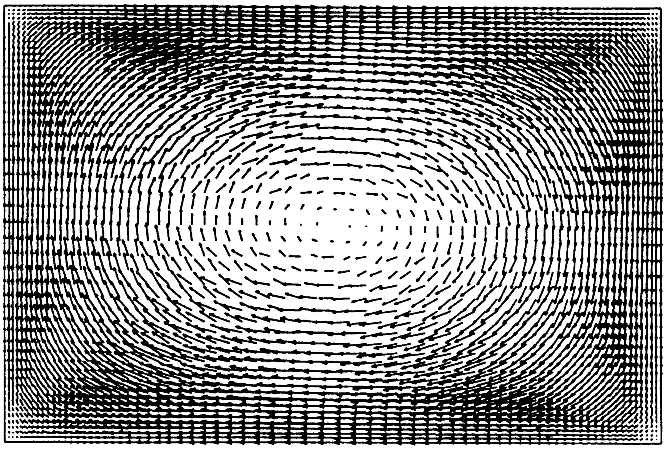
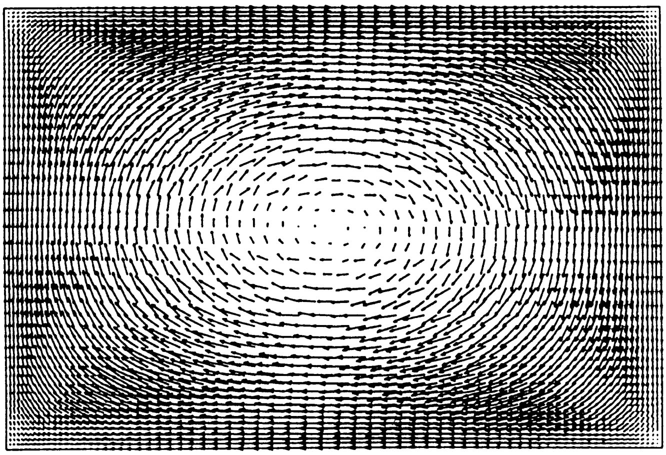


Figure 2. Finite element mesh for the test problem



(Analytical solution)



(Numerical solution)

Figure 3. Comparison of numerical solution with analytical solution (dimensionless velocity)

the problem is zero. That is to say, if the Rayleigh number of the problem is less than the critical Rayleigh number, the solution resulting from any small disturbance or perturbation converges to the trivial solution. In this case, the solution for the pore-fluid flow is zero (and, of course, symmetric) and the system is in a stable state. However, if the Rayleigh number of the problem is equal to or greater than the critical Rayleigh number, the solution resulting from any small disturbance or perturbation may lead to a non-trivial solution. In this situation, the solution for the pore-fluid flow is non-zero and the system may be in an unstable state. Since the main purpose of this study is to find out the non-trivial solution for problems having a high Rayleigh number

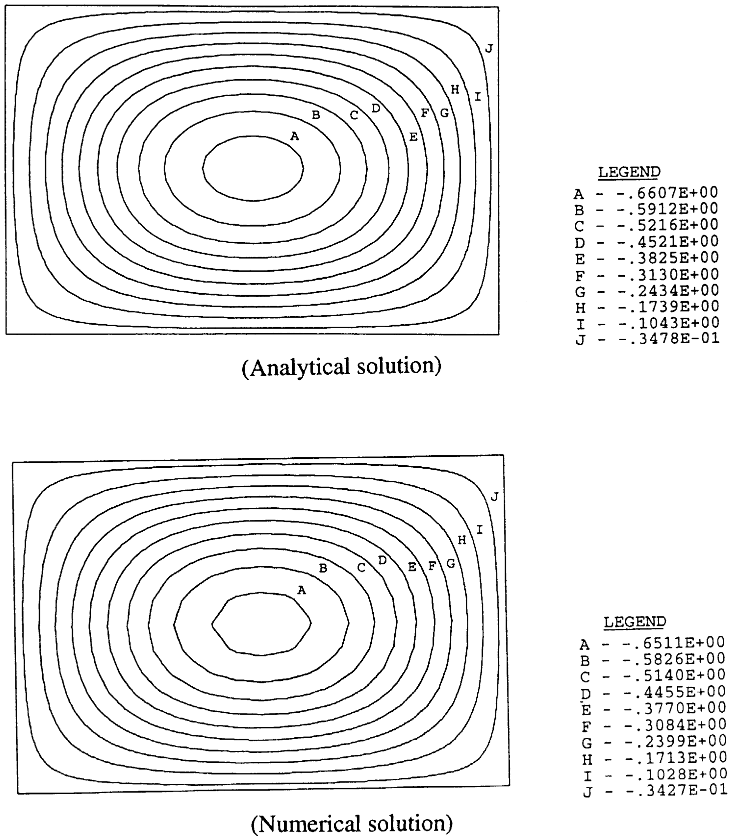


Figure 4. Comparison of numerical solution with analytical solution (dimensionless stream function)

$Ra \geqslant Ra_{\text{critical}}$, a small disturbance or perturbation needs to be applied to the system at the beginning of a computation. This is why gravity is firstly tilted a small angle away from vertical and then gradually approaches and is finally restored to vertical in the proposed progressive asymptotic approach procedure. It is the small perturbation that makes the non-trivial solution non-symmetric, even though the system considered is symmetric. In addition, as addressed in Section 3, the solution dependence on the initially-tilted small angle can be avoided by making this angle approach zero in a progressive asymptotic manner. This is one of the advantages in using the present procedure to solve Horton–Rogers–Lapwood problems.

Next, the present progressive asymptotic approach procedure is employed to investigate the effect of basin shapes on natural convection in a fluid-saturated porous medium when it is heated from below. Three different basin shapes having square, rectangular and trapezoidal geometries, which are filled with fluid-saturated porous media, are considered in the analysis. For the rectangular basin, the ratio of width to height is 1.5. For the trapezoidal basin, the ratios of top width to height and bottom width to height are 2 and 1, respectively. In order to reflect the

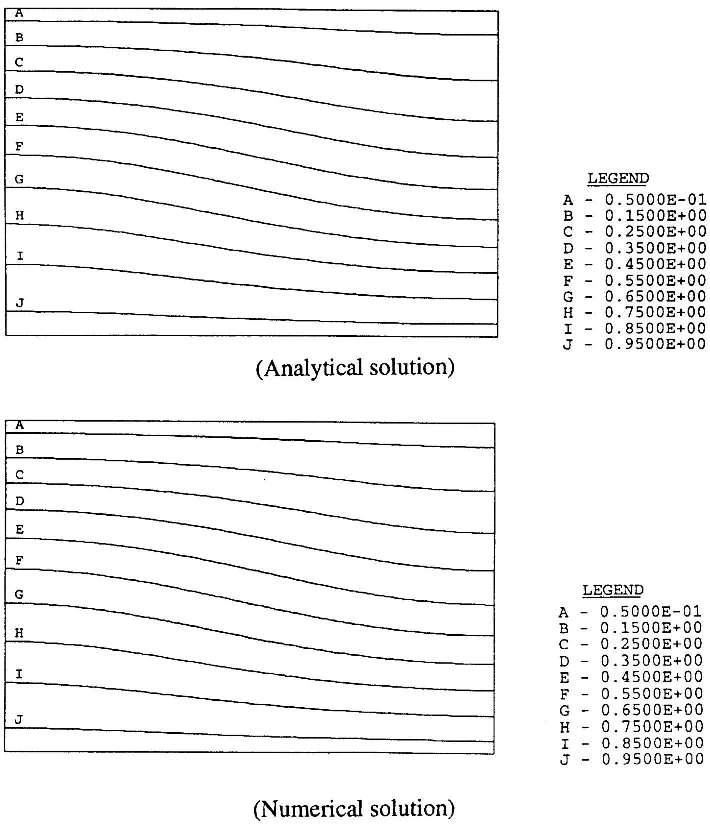
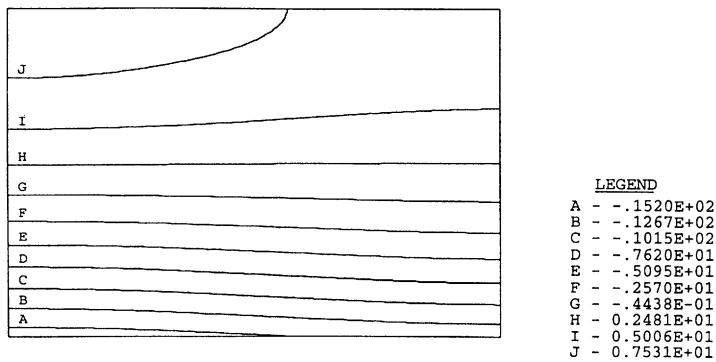


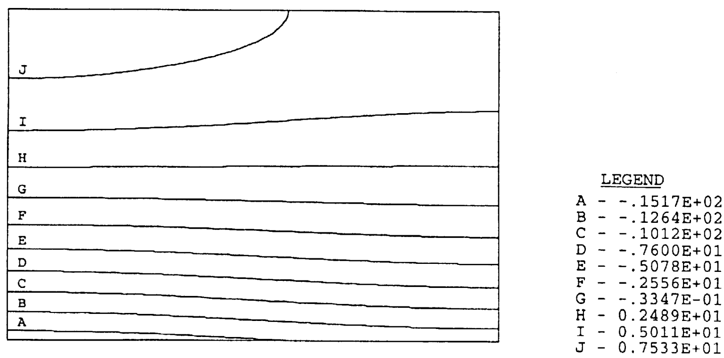
Figure 5. Comparison of numerical solution with analytical solution (dimensionless temperature)

anisotropic behaviour of the porous media, the medium permeability in the horizontal direction is assumed to be three times that in the vertical direction. As shown in Figure 7, all three basins are discretized into 576 nine-node quadrilateral elements of 2401 nodes in total. The boundary conditions of the problems are also shown in Figure 7, in which n is the normal direction of a boundary. Two Rayleigh numbers, namely $Ra = 80$ and $Ra = 400$, are used to examine the effect of the Rayleigh number on natural convection in a fluid-saturated porous medium. The same parameters as used in the above model verification examples have been used here for the progressive asymptotic approach procedure.

Figure 8 shows the dimensionless velocity distribution for the three different basins, whereas Figures 9 and 10 show the dimensionless streamline contours due to different basin shapes for $Ra = 80$ and $Ra = 400$, respectively. It is obvious that different basin shapes have a considerable effect on the patterns of convective flow in the fluid-saturated porous medium, especially in the case of higher Rayleigh numbers. Apart from notable differences in velocity distribution patterns, maximum velocity amplitudes for three different basins are also significantly different. For instance, in the case of $Ra = 80$, the maximum amplitudes of



(Analytical solution)



(Numerical solution)

Figure 6. Comparison of numerical solution with analytical solution (dimensionless pressure)

dimensionless velocities are 5·29, 8·93 and 11·66 for square, rectangular and trapezoidal basins respectively. This fact indicates that different basin shapes may affect the contaminant transport or mineralization processes in a fluid-saturated porous medium once natural convection is initiated in the medium.

6. CONCLUSIONS

Based on the asymptotic approach concept, a progressive asymptotic approach procedure associated with the finite element analysis is presented to directly solve the steady-state natural convection problems in fluid-saturated porous media when they are heated from below. Two kinds of problems, one is the modified Horton–Rogers–Lapwood problem with the gravity acceleration tilted a small angle and another is the original Horton–Rogers–Lapwood problem, need to be solved in a progressive manner. Through solving the first kind of problem progressively, an accurate non-zero velocity solution to the modified Horton–Rogers–Lapwood problem

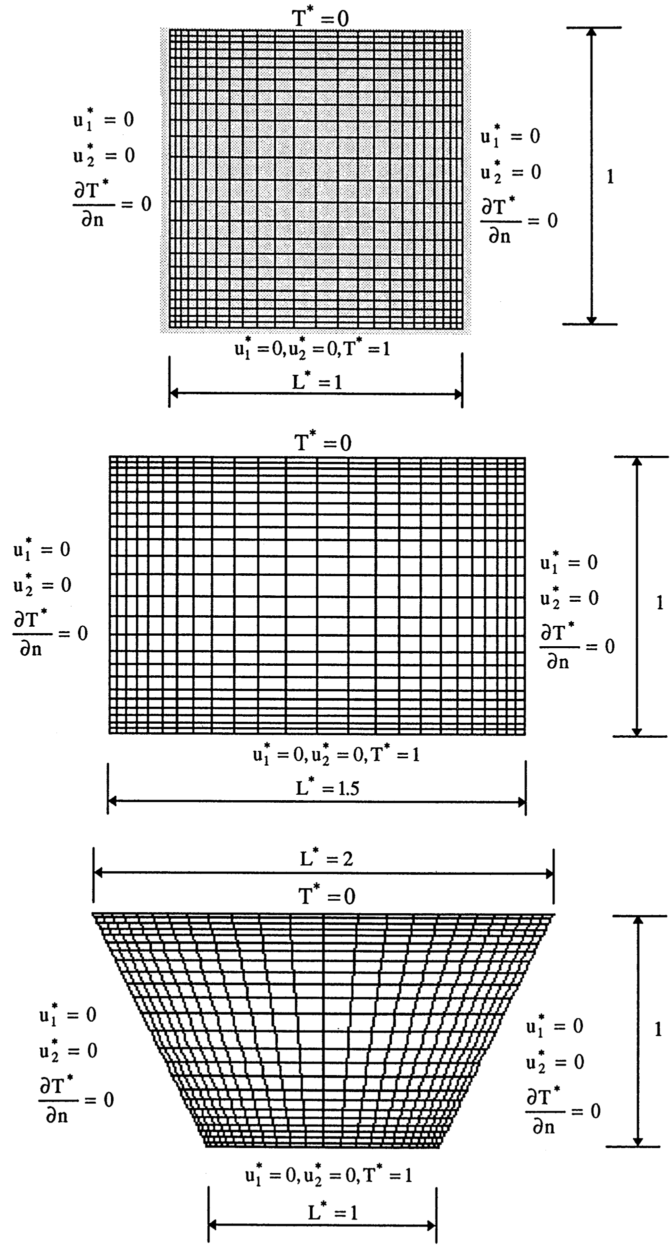
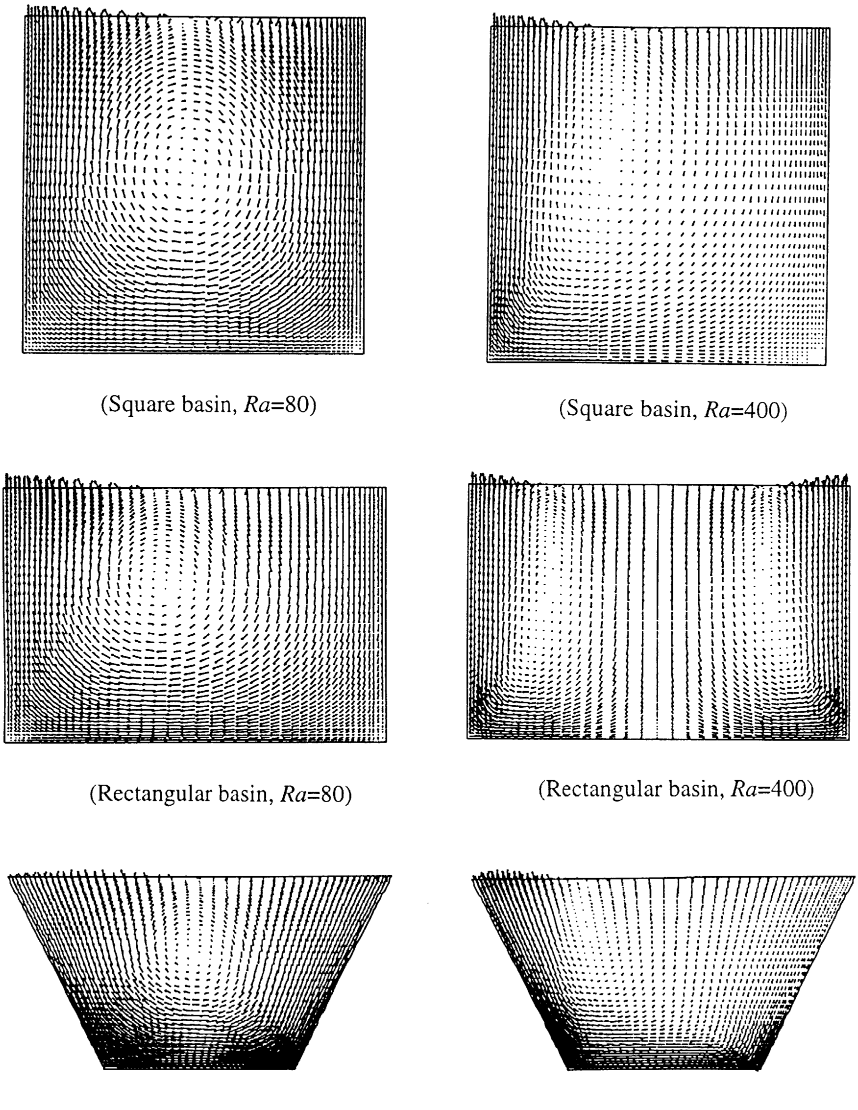


Figure 7. Finite element meshes for three different basin shapes

can be obtained. This enables a good initial prediction to be made for the solution to the original Horton–Rogers–Lapwood problem. As a result, a non-zero natural convection field can be successfully found for the original Horton–Rogers–Lapwood problem of high Rayleigh numbers when the tilted angle of the gravity acceleration approaches zero.



(Square basin, Ra=80) (Square basin, Ra=400)

(Rectangular basin, Ra=80) (Rectangular basin, Ra=400)

(Trapezoidal basin, Ra=80) (Trapezoidal basin, Ra=400)

Figure 8. Dimensionless velocity distribution for different basins

For the purpose of validating the present progressive asymptotic approach procedure, linearized analytical solutions to a benchmark problem of any rectangular geometry were derived. By comparing the numerical results with analytical ones, it has been demonstrated that the present progressive asymptotic approach procedure is effective and efficient in dealing with steady-state Horton–Rogers–Lapwood problems.

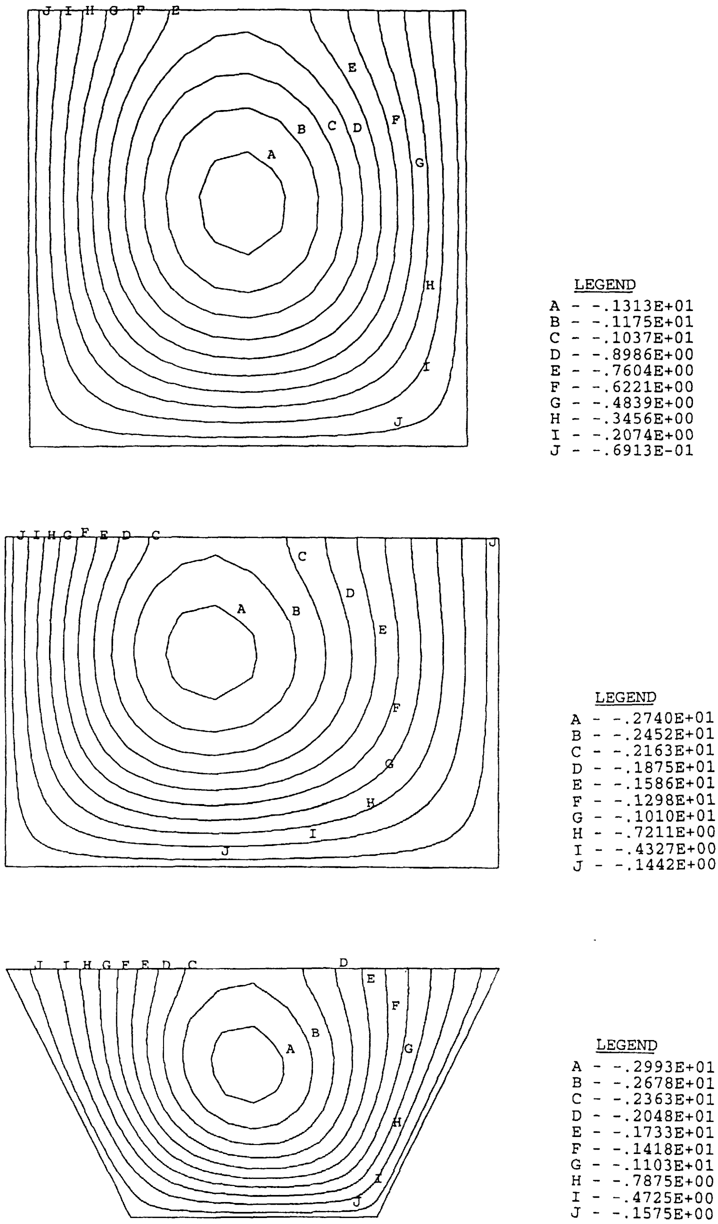


Figure 9. Dimensionless streamline contours for different basins ($Ra = 80$)

Through applying the present procedure to investigate the effect of basin shapes on natural convection in a fluid-saturated porous medium, the related numerical results have indicated that different basin shapes may affect the contaminant transport or mineralization processes once natural convection is initiated in the medium.

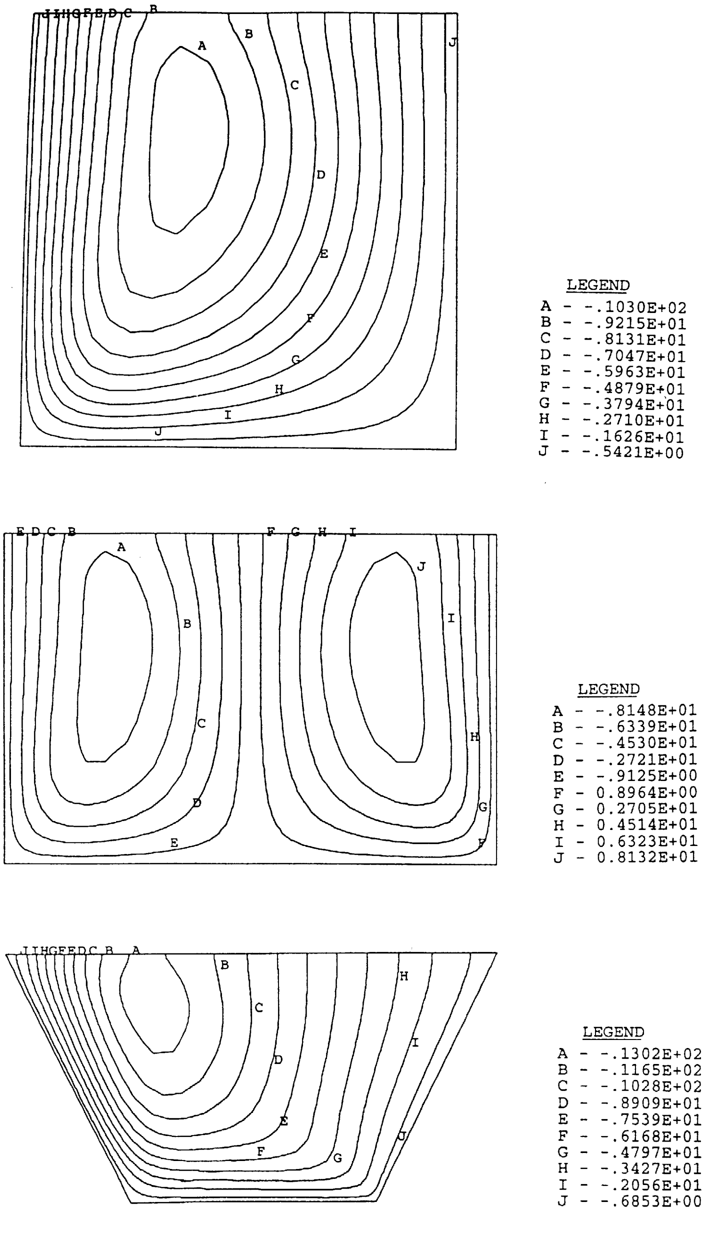


Figure 10. Dimensionless streamline contours for different basins ($Ra = 400$)

ACKNOWLEDGEMENTS

The authors are grateful to Dr. P. Hornby and the anonymous referees for their valuable comments on an early draft of this paper.

REFERENCES

1. C. W. Horton and F. T. Rogers, 'Convection currents in a porous medium', *J. Appl. Phys.*, **16,** 367–370 (1945).
2. E. R. Lapwood, 'Convection of a fluid in a porous medium', *Proc. Camb. Phil. Soc.*, **44,** 508–521 (1948).
3. D. A. Nield and A. Bejan, *Convection in Porous Media*, Springer, New York, 1992.
4. N. Joly, D. Bernard and P. Menegazzi, 'ST2D3D: A finite element program to compute stability criteria for natural convection in complex porous structures', *Numer. Heat Transfer, Part B*, **29,** 91–112 (1996).
5. O. V. Trevisan and A. Bejan, 'Mass and heat transfer by high Rayleigh number convection in a porous medium heated from below', *Int. J. Mass Heat Transfer*, **30,** 2341–2356 (1987).
6. R. D. Cook, D. S. Malkus and M. E. Plesha, *Concepts and Applications of Finite Element Analysis*, Wiley, New York, 1989.
7. C. Zhao and G. P. Steven, 'Asymptotic solutions for predicted natural frequencies of two-dimensional elastic solid vibration problems in finite element analysis', *Int. J. Num. Meth. Engng.*, **39,** 2821–2835, (1996).
8. C. Zhao and G. P. Steven, 'An asymptotic formula for correcting finite element predicted natural frequencies of membrane vibration problems,' *Commun. Numer. Meth. Engng.*, **11,** 63–73 (1996).
9. C. Zhao and G. P. Steven, 'A practical error estimator for predicted natural frequencies of two-dimensional elastodynamic problems', *Int. J. Computer-Aided Engng. Software: Engng Comput.*, **13,** 19–37 (1996).
10. O. C. Zienkiewicz, *The Finite Element Method*, McGraw-Hill, London, 1977.